On a theorem due to Birkhoff

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Abstract

The manifold M being closed and connected, we prove that every submanifold of T^*M that is Hamiltonianly isotopic to the zero-section and that is invariant by a Tonelli flow is a graph.

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1 Introduction

A famous theorem due to G. D. Birkhoff asserts that any essential invariant curve that is invariant by an area preserving twist map of the annulus is the graph of a continuous map (see [6], [13], [11], [15], [21]). Since that, a lot of attempts were made to generalize this result to higher dimensions. Under some assumptions, the authors prove that for a convex Hamiltonian of a cotangent bundle or a multidimensional positive twist map, an invariant Lagrangian manifold that is Hamiltonianly isotopic to the zero section is a graph. In general, the hypothesis is that the dynamic restricted to the invariant manifold is chain recurrent (see [2], [4], [3], [14], [5] ...). In [5], the authors ask if the result is true without such an assumption and say : "we have neither a proof nor a counterexample."
We will see that in the case of a Tonelli Hamiltonian, this hypothesis is useless. We will prove :

Theorem 1 *Let M be a compact and connected manifold. Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian and let $\mathcal{T} \subset T^*M$ be an invariant Lagrangian submanifold that is Hamiltonianly isotopic to the zero section. Then $\mathcal{T}$ is in fact a Lagrangian graph.*

The main argument to prove this theorem is the so-called weak KAM theory. This theory was built in the 90's by A. Fathi (see [10]). Another important ingredient of the proof is the use of a graph selector, or more exactly a function selector. These graph selectors give us a way to choose a pseudograph (it is a kind of discontinuous exact Lagrangian graph) in the initial Lagrangian manifold; they were firstly introduced by M. Chaperon (see [7]) and Y. Oh (see [18]); we will use the construction given by G. Paternain, L. Polterovich and K. Siburg in [20]; in this paper, a very interesting comparison is done between the graph selector and some weak KAM solutions and we will go on with this comparison. Let us mention too the preprint [19] of A. Ottolenghi and C. Viterbo where a construction is given of the so-called "geometric solution of the Hamilton-Jacobi equation." In this last paper too, the authors compare their geometric solution with other solutions, the viscosity ones (that are equal to the weak KAM ones in the autonomous case), but their result is valuable only for the time-dependent case. Curiously, we will prove that in the autonomous case too, the geometric solution corresponds to a weak KAM solution u. As u is a positive and negative weak KAM solution, we will deduce that the initial Lagrangian submanifold is in fact the graph of du.

2 A function selector

Let us recall the construction of a graph selector that is given in [20]. If $N \subset T^*M$ is a Lagrangian submanifold that is Hamiltonianly isotopic to the zero section, we can

associate with it (see [22]) what is called a generating function quadratic at infinity (gfqi) where :

DEFINITION. If $N \subset T^*M$ is Lagrangian, it admits a gfqi S if there exists a smooth function $S\ : (q,\zeta) \in M \times \mathbb{R}^N \to S(q,\zeta) \in \mathbb{R}$ such that :
• 0 is a regular value of the map $\frac{\partial S}{\partial \zeta}$; we introduce the notation : $\Sigma_S = \{(q,\zeta) \in M \times \mathbb{R}^N; \frac{\partial S}{\partial \zeta}(q,\zeta) = 0\}$; then Σ_S is a submanifold of $M \times \mathbb{R}^N$ that has the same dimension as M;
• a compact set $K \subset M \times \mathbb{R}^N$ exists so that, for every $q \in M$, the restriction of S to $(\{q\} \times \mathbb{R}^N)\backslash K$ is a non-degenerate quadratic form;
• the map $i_S\ : \Sigma_S \to T^*M$ defined by $i_S(q,\zeta) = (q, \frac{\partial S}{\partial q}(q,\zeta))$ is an embedding such that $i_S(\Sigma_S) = N$.

In this case, we have :

$$N = \{(q, d_q S(q,\zeta)); d_\zeta S(q,\zeta) = 0\}.$$

Such a generating function is used in [20] to construct a Lipschitz function $\Phi\ : M \to \mathbb{R}$ via a min-max method. This Lipschitz function Φ satisfies :
• for all $q \in M$, $\Phi(q)$ is a critical value of $S(q,.)$;
• there exists a dense open subset U_0 of M with full Lebesgue measure such that Φ is differentiable on U_0 and : $\forall q \in U_0, (q, d\Phi(q)) \in N$. Moreover : $\forall q \in U_0, \Phi(q) = S \circ i_S^{-1}(q, d\Phi(q))$.

In [20], the function is called a "graph selector", because it is used to select a part of the initial Lagrangian manifold N : $\{q, d\Phi(q)); q \in U_0\} \subset N$. But this function is more than just a graph selector : in fact, the function Φ is a means of selecting a value of S above every point $q \in M$. This is important because in the weak KAM formalism, we use continuous functions and not just discontinuous Lagrangian graphs.

We will prove in section 4 that if N is invariant by a Tonelli flow, then Φ is a C^1 function. In this case, the graph of $d\Phi$ is a submanifold of N that has the same dimension as N. As N is connected (because M is), then N is the graph of the C^0 map $d\Phi$. A classical result asserts that is the C^0 graph of $d\Phi$ is invariant by a Tonelli flow, then $d\Phi$ is Lipschitz. Being a smooth manifold that is the graph of a Lipschitz function, N is then the graph of the smooth function $d\Phi$.

3 Weak KAM theory

Except proposition 2 and its corollary, all the results of this section are proved in [10] or [1].
Let us recall that a Tonelli Hamiltonian is a C^3 function $H\ : T^*M \to \mathbb{R}$ that is :

• superlinear in the fiber : $\forall A \in \mathbb{R}, \exists B \in \mathbb{R}, \forall (q,p) \in T^*M, \|p\| \geq B \Rightarrow H(q,p) \geq A\|p\|$;
• C^2-convex in the fiber : for every $(q,p) \in T^*M$, the Hessian $\frac{\partial^2 H}{\partial p^2}$ of H in the fiber direction is positive definite as a quadratic form.
We denote the Hamiltonian flow of H by (φ_t) and the Hamiltonian vector-field by X_H. A Lagrangian function $L \;: TM \to \mathbb{R}$ is associated with H. It is defined by : $L(q,v) = \max\limits_{p \in T_q^*M} (p.v - H(q,p))$. Then L is C^2-convex and superlinear in the fiber and has the same regularity as H. We denote its Euler-Lagrange flow by (f_t). Then (φ_t) and (f_t) are conjugated by the Legendre map : $\mathcal{L} \;: (q,p) \in T^*M \to (q, \frac{\partial H}{\partial p}(q,p)) \in TM$; more precisely, we have : $\mathcal{L} \circ \varphi_t = f_t \circ \mathcal{L}$.

3.1 Domination property

3.1.1 Semigroups of Lax-Oleinik

Following A. Fathi (see [10]), we may associate two semi-groups, called Lax-Oleinik semi-groups, to any Tonelli Hamiltonian :
• the negative Lax-Oleinik semi-group $(T_t^-)_{t>0}$ is defined by :

$$\forall u \in C^0(M,\mathbb{R}), T_t^- u(q) = \min_{q' \in M} \left(u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))ds \right);$$

where the infimum is taken on the set of C^1 curves $\gamma \;: [0,t] \to M$ such that $\gamma(t) = q$.
• the positive Lax-Oleinik semi-group is defined by :

$$\forall u \in C^0(M,\mathbb{R}), T_t^+ u(q) = \max_{q' \in M} \left(u(\gamma(t)) - \int_0^t L(\gamma(s), \dot\gamma(s))ds \right);$$

where the infimum is taken on the set of C^1 curves $\gamma \;: [0,t] \to M$ such that $\gamma(0) = q$.

3.1.2 Dominated functions

If $u \in C^0(M,\mathbb{R})$ and $k \in \mathbb{R}$, we write $u \prec L + k$ and we say that u is dominated by $L + k$ if for each C^1 curve $\gamma \;: [a,b] \to M$, we have :

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot\gamma(s))ds + k(b-a).$$

Then, we have : $u \prec L + k \Leftrightarrow \forall t \geq 0, u \leq kt + T_t^- u \Leftrightarrow \forall t \geq 0, T_t^+ u - kt \leq u$.
It is proved in [10] that such a dominated function u is Lipschitz, hence differentiable almost everywhere and satisfies : $H(q, du(q)) \leq k$ at every point q of M where u is differentiable. Moreover, it is proved too that every Lipschitz function $u \;: M \to \mathbb{R}$ such that at Lebesgue almost every point q, u is differentiable and $: H(q, du(q)) \leq k$, is dominated by $L + k$.

3.1.3 weak KAM solutions and Mañé's critical value

A function $u : M \to \mathbb{R}$ is a negative (resp. positive) weak KAM solution if there exists $c \in \mathbb{R}$ such that : $\forall t > 0, T_t^- u = u - ct$ (resp. $\forall t > 0, T_t^+ u = u + ct$). Then there exists at least one positive and one negative weak K.A.M. solutions (see [10] or [1]). The constant c is unique and is called Mañé's critical value.

Many characterizations of Mañé's critical value exist. For example, it is proved in [9] that :

$$c = \inf_{u \in C^\infty(M,\mathbb{R})} \max_{q \in M} H(q, du(q)).$$

Mañé's critical value is the greatest lower bound of the set of the numbers $k \in \mathbb{R}$ for which there exists $u \in C^0(M, \mathbb{R})$ with $u \prec L + k$.

An interesting property of the weak WAM solutions is the forward (resp. backward) invariance of their pseudographs. If $u : M \to \mathbb{R}$ is a Lipschitz function, we denote the graph of du by $\mathcal{G}(du)$: $\mathcal{G}(du) = \{(q, du(q)); u \text{ is differentiable at } q\}$. Then, if u_- (resp. u_+) is a negative (resp. positive) weak KAM solution, we have : $\forall t > 0, \varphi_t(\overline{\mathcal{G}(du_-)}) \subset \mathcal{G}(du_-)$ (resp. $\varphi_{-t}(\overline{\mathcal{G}(du_+)}) \subset \mathcal{G}(du_+)$).

3.2 Mather set, Aubry set and Peierls barrier

3.2.1 Minimizing orbits and measures

Let us introduce a notation :

Notations. If $t > 0$, the function $A_t : M \times M \to \mathbb{R}$ is defined by :

$$A_t(q_0, q_1) = \inf_\gamma \int_0^t L(\gamma(s), \dot\gamma(s))ds = \min_\gamma \int_0^t L(\gamma(s), \dot\gamma(s))ds$$

where the infimum is taken on the set of C^1 curves $\gamma : [0,t] \to M$ such that $\gamma(0) = q_0$ and $\gamma(t) = q_1$. Let us recall that $\gamma_0 : [0,t] \to M$ is a critical point of A_t on the set of C^1 curves $\gamma : [0,t] \to M$ such that $\gamma(0) = q_0$ and $\gamma(t) = q_1$ if, and only if, $(\gamma, \dot\gamma)$ is an orbit piece for the Euler-Lagrange flow. We say that γ is minimizing if it achieves the minimum in the previous equality. Moreover, $\gamma : \mathbb{R} \to M$ is minimizing if its restriction to every segment is minimizing. The corresponding orbits (for (f_t) and (φ_t)) are said to be minimizing. An invariant Borel probability measure with compact support is said to be minimizing if its support is filled with minimizing orbits.

3.2.2 Mather set and conjugate weak KAM solutions

Let us introduce the Mather set :

Definition. The Mather set, denoted by $\mathcal{M}^*(H)$, is the union of the supports of the minimizing measures. The projected Mather set is $\mathcal{M}(H) = \pi(\mathcal{M}^*(H))$ where $\pi : T^*M \to M$ is the projection.

J. Mather proved in [17] that $\mathcal{M}^*(H)$ is compact, non-empty and that it is a Lipschitz graph above a compact part of the zero-section of T^*M.

A. Fathi proved in [10] that if u_- is a negative weak KAM solution, then there exists a unique positive weak KAM solution u_+ such that $u_{-|\mathcal{M}(H)} = u_{+|\mathcal{M}(H)}$. Such a pair (u_-, u_+) is called a pair of conjugate weak KAM solutions. For such a pair, we have :

- $\forall q \in \mathcal{M}(H), u_-(q) = u_+(q)$; let us denote the set of equality : $\mathcal{I}(u_-, u_+) = \{q; u_-(q) = u_+(q)\}$ by $\mathcal{I}(u_-, u_+)$; then $\mathcal{M}(H) \subset \mathcal{I}(u_-, u_+)$;
- u_- and u_+ are differentiable at every point $q \in \mathcal{I}(u_-, u_+)$; when $q \in \mathcal{M}(H)$ and $(q, p) \in \mathcal{M}^*(H)$ is its lift to $\mathcal{M}^*(H)$, then $du_-(q) = du_+(q) = p$;
- $u_+ \leq u_-$.

Moreover, if $u : M \to \mathbb{R}$ is a function such that $u \prec L + c$, then there exists a unique pair (u_-, u_+) of conjugate weak KAM solutions such that $u_{-|\mathcal{M}(H)} = u_{+|\mathcal{M}(H)} = u_{|\mathcal{M}(H)}$. In this case, we have : $u_+ \leq u \leq u_-$.

3.2.3 Aubry set

If (u_-, u_+) is a pair of conjugate weak KAM solutions, we denote by $\mathcal{I}(u_-, u_+)$ the set of equality :

$$\mathcal{I}(u_-, u_+) = \{q \in M; u_-(q) = u_+(q)\}.$$

Then $\mathcal{M}(H) \subset \mathcal{I}(u_-, u_+)$, the two functions u_- and u_+ are differentiable at every point of $\mathcal{I}(u_-, u_+)$ and their derivatives are equal on this set. We denote by $\tilde{\mathcal{I}}(u_-, u_+)$ the following lift of $\mathcal{I}(u_-, u_+)$:

$$\tilde{\mathcal{I}}(u_-, u_+) = \{(q, du_-(q)); q \in \mathcal{I}(u_-, u_+)\} = \{(q, du_+(q)); q \in \mathcal{I}(u_-, u_+)\}.$$

We have : $\mathcal{M}^*(H) \subset \tilde{\mathcal{I}}(u_-, u_+)$ and it is proved in [10] that $\tilde{\mathcal{I}}(u_-, u_+)$ is a Lipschitz graph above $\mathcal{I}(u_-, u_+)$.
The Aubry set is defined by :

$$\mathcal{A}^*(H) = \bigcap \tilde{\mathcal{I}}(u_-, u_+)$$

where the intersection is taken on the set of pairs (u_-, u_+) of conjugate weak KAM solutions. The projected Aubry set is : $\mathcal{A}(H) = \pi(\mathcal{A}^*(H))$. Then $\mathcal{A}^*(H)$ is a Lipschitz graph above $\mathcal{A}(H)$ that is closed, non-empty and invariant.

The Peierls barrier $h : M\times M \to \mathbb{R}$ is defined by : $h(q_1,q_2)=\liminf_{T\to+\infty}(A_T(q_1,q_2)+cT)$. It is proved in [10] that h is Lipschitz and that the previous $\liminf$ is in fact a true limit, and even an uniform limit. Moreover, we have :

- for every $q\in M$: $h(q,q)\geq 0$;
- if $u\prec L+c$, then : $\forall q_1,q_2\in M, u(q_2)-u(q_1)\leq h(q_1,q_2)$;
- $\forall q\in M, q\in\mathcal{A}(H)\Leftrightarrow h(q,q)=0$.

We deduce easily that q belongs to $\mathcal{A}(H)$ if and only if there exists a sequence $(t_n)\in\mathbb{R}_+$ tending to $+\infty$ and a sequence of curves $\gamma_n : [0,t_n]\to M$ such that $\lim_{n\to\infty}\gamma_n(0)=\lim_{n\to\infty}\gamma_n(t_n)=q_n$ and $\lim_{n\to\infty}\int_0^{t_n}(L(\gamma_n,\dot\gamma_n)+c)\leq 0$. In this case, the last limit is equal to 0 and $\lim_{n\to\infty}(\gamma_n(0),\frac{\partial L}{\partial v}(\gamma_n(0),\dot\gamma_n(0)))=\lim_{n\to\infty}(\gamma_n(t_n),\frac{\partial L}{\partial v}(\gamma_n(t_n),\dot\gamma_n(t_n)))$ is the point of $\mathcal{A}^*(H)$ that is above q.

3.3 More on the weak KAM theory

In [10], Albert Fathi proves that a function that is a positive and negative weak KAM solution is $C^{1,1}$. Let us now give a result that may be useful to prove that some functions are positive and negative weak KAM solutions.

Proposition 2 *Let $u : M\to\mathbb{R}$ be a dominated function : $u\prec L+c$ and (u_-,u_+) the pair of conjugate weak KAM solutions such that $u=u_-=u_+$. Then :*

- *if : for almost $q\in M, \exists q_0\in\mathcal{A}(H), u(q_0)-u(q)\geq h(q,q_0)$, then $u=u_+$;*
- *if : for almost $q\in M, \exists q_0\in\mathcal{A}(H), u(q)-u(q_0)\geq h(q_0,q)$, then $u=u_-$.*

PROOF We only prove the first point, the second one being similar. We know that $u_+\leq u\leq u_-$ and that : $\forall q_0\in\mathcal{A}(H), u(q_0)=u_-(q_0)=u_+(q_0)$.
Let us now consider $q\in M$ such that there exists $q_0\in\mathcal{A}(H)$ such that $u(q_0)-u(q)\geq h(q,q_0)$. As $u_+\leq u$ and $u(q_0)=u_+(q_0)$, we have : $u(q_0)-u(q)\leq u_+(q_0)-u_+(q)$. As u_+ is a weak KAM solution, it is dominated by $L+c$. We have then :

$$h(q,q_0)\leq u(q_0)-u(q)\leq u_+(q_0)-u_+(q)\leq h(q,q_0).$$

We deduce that $u(q_0)-u(q)=u_+(q_0)-u_+(q)$ and then $u(q)=u_+(q)$. The two functions u and u_+ are continuous and equal almost everywhere, they are then equal everywhere. □

Corollary 3 *Let $u : M \to \mathbb{R}$ be a dominated function : $u \prec L + c$ such that : for almost $q \in M, \exists q_1, q_2 \in \mathcal{A}(H), u(q_1) - u(q) \geq h(q, q_1)$ and $u(q) - u(q_2) \geq h(q_2, q)$. Then u is $C^{1,1}$ and the graph of du is invariant by the Hamiltonian flow.*

Proof Let $u : M \to \mathbb{R}$ satisfy the hypotheses of the corollary. We deduce from proposition 2 that u is a positive and negative weak KAM solution. Hence, u is $C^{1,1}$. We have then : $\forall t > 0, \mathcal{G}(du) = \overline{\mathcal{G}(dT_t^- u)} \subset \varphi_t(\mathcal{G}(du))$ and $\mathcal{G}(du) = \overline{\mathcal{G}(dT_t^+ u)} \subset \varphi_{-t}(\mathcal{G}(du)))$. Hence the graph $\mathcal{G}(du)$ is invariant. ∎

4 Proof of theorem 1

Two submanifolds of T^*M that are Hamiltonianly isotopic to the zero section have a non-empty intersection (see [16]). Let us now consider a submanifold N of T^*M that is Hamiltonianly isotopic to the zero section and that is invariant by the Tonelli flow of H. Then as N is an invariant Lagrangian submanifold, there exists $k \in \mathbb{R}$ such that $N \subset \{H = k\}$. Moreover, the intersection of N with any $\mathcal{G}(du)$ for $u \in C^2(M, \mathbb{R})$ is non-empty because the two manifolds are Hamiltonianly isotopic to the zero-section. We have seen that Mañé's critical value is given by : $c = \inf_{u \in C^\infty(M,\mathbb{R})} \max_{q \in M} H(q, du(q))$. Then we have : $k \leq c$.

We assume now that N is a submanifold that is Hamiltonianly isotopic to the zero section and that is invariant under the Tonelli flow of H. We have noticed that there exists $k \leq c$ such that $N \subset \{H = k\}$.
Moreover, we have built in section 2 a generating function S and a function selector Φ. There exists a dense open subset U_0 of M with full Lebesgue measure such that Φ is differentiable on U_0 and : $\forall q \in U_0, (q, d\Phi(q)) \in N$. Hence, at Lebesgue almost every point, we have : $H(q, d\Phi(q)) \leq k$. We have seen that this implies : $\Phi \prec L + k$. As $k \leq c$ and c is the greatest lower bound of the set of the numbers $k \in \mathbb{R}$ for which there exists $u \in C^0(M, \mathbb{R})$ with $u \prec L + k$, we deduce that $k = c$.

4.1 Place of the Aubry set

The beginning of this proposition is proved in [20].

Proposition 4 *If N is a submanifold that is Hamiltonianly isotopic to the zero section and that is invariant under the Tonelli flow of H, if $\Phi : M \to \mathbb{R}$ is the associated function selector, then at every $q \in \mathcal{A}(H)$, Φ is differentiable, $(q, d\phi(q)) \in N$ and $\Phi(q) = S \circ i_S^{-1}(q, d\Phi(q))$.*

We need a lemma :

Lemma 5 *Let $f : U \to \mathbb{R}$ be a Lipschitz function defined on a open subset U of $\mathbb{R}^d$ and let $U_0 \subset U$ be a subset with full Lebesgue measure such that f is differentiable at every point of U_0. We introduce a notation : if $q \in U$, $K_f(q)$ is the set of all the limits $\lim\limits_{n\to\infty} df(q_n)$ where $q_n \in U_0$, $\lim\limits_{n\to\infty} q_n = q$ and $C_f(q)$ is the convex hull of $K_f(q)$. Then, at every point $q \in U$ where f is differentiable, we have : $df(q) \in C_f(q)$.*

Proof This lemma is proved in [12]. A more general result is proved in [8] too. Let us give an idea of a simple proof. Using Fubini theorem, we obtain for every $v \in \mathbb{R}^d$ a sequence of vectors (v_n) converging to v and a decreasing sequence (t_n) tending to 0 such that : $df(q)v = \lim\limits_{n\to\infty} \dfrac{1}{t_n}\displaystyle\int_0^{t_n} df(q+sv_n).v_n ds$ where for Lebesgue almost every point $t \in [0, t_n]$, we have : $q + tv_n \in U_0$. Then, for every $v \in \mathbb{R}^d$, we find $p_v \in C_f(q)$ such that $df(q)v = p_v(q)$; as $C_f(q)$ is convex and compact, using Hahn-Banach theorem, we deduce : $df(q) \in C_f(q)$. □

As $\Phi \prec L + c$, there exists a pair (u_-, u_+) of conjugate weak KAM solutions such that $u_{-|\mathcal{M}(H)} = u_{+|\mathcal{M}(H)} = \Phi_{|\mathcal{M}(H)}$ and we have : $u_+ \le \Phi \le u_-$. As u_+ and u_- are differentiable on $\mathcal{A}(H)$ and as $u_{-|\mathcal{A}(H)} = u_{+|\mathcal{A}(H)} = \Phi_{|\mathcal{A}(H)}$ and $du_{-|\mathcal{A}(H)} = du_{+|\mathcal{A}(H)}$, we deduce that Φ is differentiable on $\mathcal{A}(H)$ and that : $\forall q \in \mathcal{A}(H), (q, d\Phi(q)) = (q, du_-(q)) \in \mathcal{A}^*(H)$. We cannot conclude that $\mathcal{A}^*(H) \subset N$ because we don't know if $\mathcal{A}(H) \subset U_0$. We use then lemma 5 (we work in a chart). Let $q_0 \in \mathcal{A}(H)$ be an element of the projected Aubry set. We deduce from the lemma that : $d\Phi(q_0) \in C_\Phi(q_0)$. Moreover, $d\Phi(q_0) \in T^*_{q_0}M \cap \{H = c\}$ and $T^*_{q_0}M \cap \{H = c\}$ is the set of the extremal points of the convex set $T^*_{q_0}M \cap \{H \le c\}$ and this last set contains $C_\Phi(q_0)$. Then $d\Phi(q_0)$ is an extremal point of $C_\Phi(q_0)$ and then $d\Phi(q_0)$ belongs to $K_\Phi(q_0)$. It means that there exists a sequence (q_n) of points of U_0 that converge to q_0 so that : $d\Phi(q_0) = \lim\limits_{n\to\infty} d\Phi(q_n)$. We deduce that $(q_0, d\Phi(q_0)) \in N$. Moreover, as Φ, S and i_S are continuous :

$$\Phi(q_0) = \lim_{n\to\infty} \Phi(q_n) = \lim_{n\to\infty} S \circ i_S^{-1}(q_n, d\Phi(q_n)) = S \circ i_S^{-1}(q_0, d\Phi(q_0)).$$

4.2 Place of the non-wandering set

Proposition 6 *If N is a submanifold that is Hamiltonianly isotopic to the zero section and that is invariant under the Tonelli flow of H, we have : $\Omega(\varphi_{t|N}) \subset \mathcal{A}^*(H)$.*

Let us explain why the non-wandering set $\Omega(\varphi_{t|N})$ of the Hamiltonian flow restricted to N is in the Aubry set for H. A similar argument is given is [20]. We have noticed that $N \subset \{H = c\}$. Let $(q, p) \in \Omega(\varphi_{t|N})$. Then there exist a sequence (q_n, p_n) of points of N converging to (q, p) and a sequence (t_n) in $\mathbb{R}_+$ tending to $+\infty$ such that : $\lim\limits_{n\to\infty} \varphi_{t_n}(q_n, p_n) = (q, p)$. Let us introduce the notation : $(q_n(t), p_n(t)) = \varphi_t(q_n, p_n)$.

As N is Hamiltonianly isotopic to the zero-section, it is exact Lagrangian and then : $\lim_{n\to\infty}\int_0^{t_n} p_n(t)\dot{q}_n(t)dt = 0$. As $(q_n(t), p_n(t))$ is an orbit, we have : $\dot{q}_n = \frac{\partial H}{\partial p}(q_n, p_n)$, and then : $p_n.\dot{q}_n = L(q_n, \dot{q}_n) + H(q_n, p_n) = L(q_n, \dot{q}_n) + c$. Finally, we have :

$$\lim_{n\to\infty}\int_0^{t_n}(L(q_n(t), \dot{q}_n(t)) + c)dt = 0.$$

As $\lim_{n\to\infty} q_n(0) = q$ and $\lim_{n\to\infty} q_n(t_n) = q$, we deduce that :

$$(q, p) = \lim_{n\to\infty}(q_n(0), p_n(0)) = \lim_{n\to\infty}(q_n(0), \frac{\partial L}{\partial v}(q_n(0), \dot{q}_n(0))) \in \mathcal{A}^*(H).$$

Hence we have proved :

$$\Omega(\varphi_{t|N}) \subset \mathcal{A}^*(H).$$

4.3 Comparison between h and Φ

Let us now prove that Φ satisfies the hypotheses of corollary 3.

Proposition 7 *For all $q \in U_0$, there exists $q_1, q_2 \in \mathcal{A}(H)$ such that : $\Phi(q_1) - \Phi(q) \geq h(q, q_1)$ and $\Phi(q) - \Phi(q_2) \geq h(q_2, q)$.*

We consider $q \in U_0$. Then $(q, d\Phi(q)) \in N$ and $\Phi(q) = S \circ i_S^{-1}(q, d\Phi(q))$. Then the α and ω limit sets of $(q, d\Phi(q))$ are non-empty. There exist $(q_1, p_1) \in \omega(q, d\Phi(q))$ and $(q_2, p_2) \in \alpha(q, d\Phi(q))$. These points being non-wandering and in N, we have noticed that they belong to $\mathcal{A}^*(H)$: $(q_i, p_i) = (q_i, d\Phi(q_i)) \in \mathcal{A}^*(H)$ and that : $\Phi(q_i) = S \circ i_S^{-1}(q_i, d\Phi(q_i))$. As they belong to the α/ω limit set, there exist two sequences $(t_n) \in \mathbb{R}_+$ and $(\tau_n) \in \mathbb{R}_+$ tending to $+\infty$ so that :

$$\lim_{n\to\infty}\varphi_{t_n}(q, d\Phi(q)) = (q_1, p_1) \quad \text{and} \quad \lim_{n\to\infty}\varphi_{-\tau_n}(q, d\Phi(q)) = (q_2, p_2).$$

We use the following notation : $\varphi_t(q, d\Phi(q)) = (q(t), p(t))$ and we compute :

$$\Phi(q_1)-\Phi(q) = S\circ i_S^{-1}(q_1, d\Phi(q_1))-S\circ i_S^{-1}(q, d\Phi(q)) = \lim_{n\to\infty} S\circ i_S^{-1}\circ\varphi_{t_n}(q, d\Phi(q))-S\circ i_S^{-1}(q, d\Phi(q))$$

We have :

$$S \circ i_S^{-1} \circ \varphi_{t_n}(q, d\Phi(q)) - S \circ i_S^{-1}(q, d\Phi(q)) = S \circ i_S^{-1}(q(t_n), p(t_n)) - S \circ i_S^{-1}(q(0), p(0))$$

where : $i_s(q, \zeta) = (q, \frac{\partial S}{\partial q}(q, \zeta))$ and on Σ_S : $\frac{\partial S}{\partial \zeta} = 0$. Hence $i_s^{-1}(q, p) = (q, \beta(q, p))$ and : $\forall(\delta q, \delta p) \in T_{(q,p)}N$: $d(S \circ i_S^{-1})(q, p)(\delta q, \delta p) = \frac{\partial S}{\partial q}(i_S^{-1}(q, p))\delta q$. Then :

$$S\circ i_S^{-1}\circ\varphi_{t_n}(q, d\Phi(q))-S\circ i_S^{-1}(q, d\Phi(q)) = \int_0^{t_n}\frac{\partial S}{\partial q}(i_S^{-1}(q(t), p(t)))\dot{q}(t)dt = \int_0^{t_n} p(t)\dot{q}(t)dt$$

As $(q(t), p(t))$ is an orbit, we have : $\dot{q} = \frac{\partial H}{\partial p}(q, p)$, and then : $p.\dot{q} = L(q, \dot{q}) + H(q, p) = L(q, \dot{q}) + c$. Finally, we have :

$$S \circ i_S^{-1} \circ \varphi_{t_n}(q, d\Phi(q)) - S \circ i_S^{-1}(q, d\Phi(q)) = \int_0^{t_n} (L(q(t), \dot{q}(t)) + c)dt.$$

We deduce that :

$$h(q, q_1) \leq \lim_{n\to\infty} S \circ i_S^{-1} \circ \varphi_{t_n}(q, d\Phi(q)) - S \circ i_S^{-1}(q, d\Phi(q)) = \Phi(q_1) - \Phi(q).$$

In a similar way, we obtain : $h(q_2, q) \leq \Phi(q) - \Phi(q_2)$.

4.4 Conclusion

We deduce from this and from corollary 3 that Φ is $C^{1,1}$ and that $\mathcal{G}(d\Phi)$ is invariant by the flow.

Let us now summarize what we did :

- we have found a dense part $\mathcal{G}(d\Phi_{|U_0})$ of $\mathcal{G}(d\Phi)$ that is a subset of the closed manifold N. Hence $\mathcal{G}(d\Phi) \subset N$;
- hence $\mathcal{G}(d\Phi)$ is a closed submanifold of N that has the same dimension as N; N being connected, we deduce that $\mathcal{G}(d\Phi) = N$ is a graph.

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